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Exact solution of an exclusion process with three classes of particles and vacancies

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Abstract. We present an exact solution for an asymmetric exclusion process on a ring with three classes of particles and vacancies. Using a matrix-product Ansatz, we find exact expressions for the weights of the configurations in the stationary state. The solution involves tensor products of quadratic algebras.

1. Introduction

The one-dimensional asymmetric simple exclusion process (ASEP) has been extensively studied in mathematical and physical literature as one of the simplest models for non-equilibrium statistical mechanics [1–4]. The ASEP is a model of particles diffusing on a lattice driven by an external field with hard-core exclusion. A variety of different phenomena can be described by the exclusion process, for instance superionic conductors [5], traffic flows [6] and interface growth [7]. Exact results have been obtained for the one-dimensional exclusion process with the help of various methods such as the Bethe Ansatz [8–11], and, more recently, a matrix-product Ansatz (see, for example, [12]).

The matrix Ansatz has led to new exact results concerning the stationary state of various models. Originally developed for the study of directed animals on a lattice [13], this method has been successfully applied to the exclusion process with open boundaries [14]. It was then extended to study shocks in systems with second-class particles [15], time-discrete dynamics [16–21], and to calculate diffusion constants [22]. The algebras involved have led to interesting representation problems [23].

Models with more than two classes of particles have hardly been investigated [24–26]. Here, we study an exclusion model with vacancies and three classes of particles on a ring. Up to now, it was not known whether the matrix Ansatz could be used to construct an exact solution of this model. In this paper, we shall give an exact expression for the stationary state of our model by using a suitable matrix Ansatz that involves tensor products of quadratic algebras. Apart from the question of as to how far the matrix Ansatz can be used, the model is interesting in itself, because it is suitable for a detailed study of shocks [27]. In section 2, we define the model and explain what the matrix Ansatz is. In section 3 we recall the matrix solution for the asymmetric exclusion process with second-class particles. In section 4, we give an explicit representation of the operators that allow one to calculate the stationary state of our model and present a proof of our solution. We also give an explicit solution for the stationary state without using any representations. We then discuss an algorithm that allows one to obtain exact properties of the stationary state for large systems by using a computer. The concluding section discusses our results and some generalizations. The appendices contain details of the proof and certain algebraic properties leading to recursion relations between systems of different sizes.

2. Definition of the model

We consider a periodic one-dimensional lattice of L sites. Each site of the lattice is either empty or occupied by a single particle that can be of type 1, 2 or 3. For reasons that will become apparent later on, we say that empty sites are occupied by holes (or vacancies) and we shall call holes *particles of the fourth type*. We denote the number of particles of type k in the system by n_k, where $k = 1, 2, 3, 4$. The state of a site i is specified by a discrete variable τ_i that takes four possible values:

$$\tau_i = 1, 2, 3 \text{ or } 4 \text{ if site } i \text{ is occupied by a particle of type } 1, 2, 3 \text{ or } 4. \tag{1}$$

The dynamics of the system is given by certain transition rules. During an infinitesimal time step dt, the following processes take place on a bond $(i, i+1)$ with probability dt:

$$\begin{aligned} &12 \to 21 \text{ with rate } 1\\ &13 \to 31 \text{ with rate } 1\\ &14 \to 41 \text{ with rate } 1\\ &23 \to 32 \text{ with rate } 1\\ &24 \to 42 \text{ with rate } 1\\ &34 \to 43 \text{ with rate } 1. \end{aligned} \tag{2}$$

All other transitions are forbidden. Obviously, the dynamics conserves the number of particles and one has $\sum_{k=1}^{4} n_k = L$. It is also clear from these rules that particles of type n can 'overtake' particles of type m only if $n < m$. The transition rules therefore induce a hierarchy among the particles.

In the literature, particles of type $k = 1, 2, 3$ are named *first-*, *second-* and *third-class particles*. The model defined in (2) is 'the totally asymmetric exclusion process (TASEP) with three classes of particles and holes'.

Note that a first-class particle always behaves in the same way in regards to all the other particles, whereas a third-class particle, for example, behaves like a first-class particle with respect to the holes, but as a hole with respect to the second- and first-class particles.

The rules given in (2) are translationally invariant. Using this property, we can decide, without loss of generality, that a particle of the third class occupies the site number L and we enumerate the different configurations of the system. The total number of configurations N_{tot} is given by

$$N_{\text{tot}} = \frac{(L-1)!}{n_1!n_2!(n_3-1)!n_4!}. \tag{3}$$

The dynamics of the system can be encoded in a Markov matrix M of size $N_{\text{tot}} \times N_{\text{tot}}$. The coefficient $M(\mathcal{C}, \mathcal{C}')$ of this matrix represents the rate of transition from a configuration $\mathcal{C}$ to a different configuration $\mathcal{C}'$; $M(\mathcal{C}, \mathcal{C})$ is the exit rate from a given configuration $\mathcal{C}$. Due to the local structure of the rules (2), M can be written as a sum of local operators that represent the transitions that take place at a bond $(i, i+1)$

$$M = \sum_{i=1}^{L} m_{i,i+1}. \tag{4}$$

An explicit expression for the matrices $m_{i,i+1}$ is given in the appendix.

In the long-time limit, the process reaches a stationary state in which each configuration $\mathcal{C}$ of the system has a stationary probability $p(\mathcal{C})$. The stationary state exists and is unique. This follows from the fact that the rules (2) define an irreducible Markov process, i.e. any given configuration can evolve to any other one. The properties of the stationary state can be determined once the probabilities $p(\mathcal{C})$ are known for all $\mathcal{C}$. In equilibrium statistical mechanics these numbers are given by the Boltzmann factor, but in our model there is *a priori* no method to calculate these quantities: one has to solve the stationary master equation

$$\sum_{\mathcal{C}'} M(\mathcal{C}, \mathcal{C}') p(\mathcal{C}') = 0. \tag{5}$$

This is a system of N_{tot} coupled linear equations whose complexity grows exponentially with the size, L, of the system.

The matrix Ansatz [14] consists of solving system (5) by writing the probabilities $p(\mathcal{C})$ as traces of products of four non-commuting operators, A_1, A_2, A_3 and A_4, each representing one type of particle:

$$p(\mathcal{C}) = \frac{1}{Z} \operatorname{Tr}(A_{\tau_1} \ldots A_{\tau_L}) \tag{6}$$

where A_{τ_i} is equal to A_k if site i is occupied by a particle of type k ($k = 1, 2, 3, 4$) in configuration $\mathcal{C}$. The constant Z is an overall normalization factor, that depends on L and n_k; it ensures that $\sum p(\mathcal{C}) = 1$. Hence, the matrix Ansatz expresses the stationary weights as traces over the algebra generated by the operators A_k. The calculation of the stationary weights is therefore reduced to the problem of finding this algebra (either an explicit representation, or an abstract characterization via generators and relations) and computing traces over it.

In section 4, we shall present explicit A_k operators ($k = 1, 2, 3, 4$) and prove that the weights $p(\mathcal{C})$ constructed from (6) are solutions of the master equation (5). Certain properties of the solution of the model with only first- and second-class particles (i.e. in our case $n_3 = 0$) are useful for constructing the A_k operators. Therefore, we review this solution in the following section.

3. Matrix solution of the TASEP with first- and second-class particles

Suppose initially, that there are only first-class particles and holes on the ring. In this case, the steady state is such that all configurations $\mathcal{C}$ have the same weight [28]. Assuming that the last site is always occupied by a first-class particle, we obtain that the total number of configurations will be $\frac{(L-1)!}{(L-n_1)!(n_1-1)!}$ and each configuration has a stationary probability equal to

$$p(\mathcal{C}) = \frac{(n_1 - 1)!(L - n_1)!}{(L - 1)!}. \tag{7}$$

In this simple case, a matrix Ansatz is not needed (one can choose the matrices representing particles and holes to be both equal to the scalar 1).

We now consider the model defined in (2) without particles of type 3 (i.e. $n_3 = 0$). There are n_1 first-class particles and n_2 second-class particles. For this model, the stationary probability is non-uniform and was obtained in [15] from a matrix product Ansatz. Following [15], we denote by D, E and A the operators that represent particles of type 1, 2 and holes, respectively. The numbers $p(\mathcal{C})$ obtained from expression (6) are the stationary probabilities of the exclusion process with first- and second-class particles, if the three operators E, D and A generate the quadratic algebra defined by the relations

$$\begin{aligned} DE &= D + E \\ DA &= A \\ AE &= E. \end{aligned} \tag{8}$$

It is convenient to work with an explicit representation of the algebra (8). A particularly useful choice is

$$\begin{aligned} D &= \begin{pmatrix} 1 & 1 & 0 & 0 & . & . \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & . & \\ . & & & & . & . \\ . & & & & & . \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & . & . \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ . & & & . & . & \\ . & & & & . & . \end{pmatrix} \\ A &= |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ . & & & . & . & \\ . & & & & . & . \end{pmatrix}. \end{aligned} \tag{9}$$

The operators D and E are represented by matrices that act on an infinite-dimensional space with denumerable basis $(|1\rangle, |2\rangle, \ldots |n\rangle, \ldots)$. The operator A is a projector of rank 1 on the first element of the basis and has a finite trace. This ensures that any expression of the type (6) is finite.

Using the algebraic rules (8) or the explicit representation (9), all stationary probabilities are determined. In order to calculate physical quantities such as density profiles, or average local currents, we must know the constant Z, which plays a role analogous to that of the partition function. The expression for Z is simple if there is only one second-class particle in the system, always located on the last site. In that case, one obtains [15]

$$Z = \frac{1}{L}\binom{L}{n_1}\binom{L}{n_1+1}.$$

When the density of second-class particles is finite, asymptotic formulae for Z are derived for systems of large size, using a grand canonical formalism [15].

4. Matrix solution of the TASEP with first-, second- and third-class particles

4.1. Explicit forms of the matrices

We shall show that the stationary weights, solutions of the master equation (5), can be calculated via Ansatz (6), from the following four operators:

$$A_1 = \begin{pmatrix} D & 0 & E & 0 & 0 & . \\ 0 & D & 0 & E & 0 & . \\ 0 & 0 & D & 0 & E & . \\ 0 & 0 & 0 & D & 0 & . \\ 0 & 0 & 0 & 0 & D & . \\ . & . & . & . & . & . \end{pmatrix} \qquad A_2 = \begin{pmatrix} D & -E & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . \end{pmatrix}$$
$$A_3 = \begin{pmatrix} E & 0 & 0 & 0 & 0 & . \\ D & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . \end{pmatrix} A_4 = \begin{pmatrix} E & 0 & 0 & 0 & 0 & . \\ 0 & E & 0 & 0 & 0 & . \\ D & 0 & E & 0 & 0 & . \\ 0 & D & 0 & E & 0 & . \\ 0 & 0 & D & 0 & E & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{10}$$

All matrices are infinite-dimensional and their coefficients are themselves the infinite-dimensional operators D and E of (9) which satisfy $DE = D + E$ and do not commute with each other (i.e. scalar representations of D and E are excluded). Another way to look at the operators given in (10) is to consider them as matrices operating on an infinite-dimensional space, with non-commutative elements. The operators A_2 and A_3 have only two non-zero elements and the following relation holds:

$$A_2 A_3 = \begin{pmatrix} A & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{11}$$

Here A is the rank-one projector of (9). Before we prove that the stationary probabilities given in terms of A_i solve the master equation (5), we have to ensure that they are finite. This is not obvious because none of the operators given in (10) has a finite trace.

4.2. Proof of the finiteness of the Ansatz

We rewrite expression (6) for the stationary weights as follows:

$$p(\mathcal{C}) = \frac{1}{Z}\,\mathrm{Tr}(A_{\tau_1}\dots A_{\tau_L}) = \frac{1}{Z}\,\mathrm{Tr}(Y A_2 X A_3 T) \tag{12}$$

where Y and T are products of A_k $(k = 1, 2, 3, 4)$, and X is a product of p $(p \leqslant L - 2)$ matrices A_1 and A_4 only. Such a factorization is possible: the term $A_2 X A_3$ starts from the furthermost (proceeding from left to right) factor A_2 in $(A_{\tau_1}\dots A_{\tau_L})$, and ends when an A_3 appears for the first time after this A_2. Such a factor A_3 always exists since $A_{\tau_L} = A_3$.

We now prove that the matrix $A_2 X A_3$, where X is a product of p factors A_1 and A_4, can only have non-zero elements in its first $(p + 2)$ lines or columns. We shall say, in such a case, that $A_2 X A_3$ is 'of finite size $(p + 2)$'.

The operators A_0 and A_1 have two invariant subspaces, the subspace generated by the odd vectors of the basis $(|1\rangle, |3\rangle, \dots |2n + 1\rangle, \dots)$ and the subspace generated by

$(|2\rangle, |4\rangle, \dots |2n\rangle, \dots)$. The action of A_1 (and that of A_4) on both invariant subspaces is the same. Therefore, the product X will be represented by the following matrix:

$$X = \begin{pmatrix} \chi & 0 & \star & 0 & & \\ 0 & \chi & 0 & \star & . & . \\ \star & 0 & \star & 0 & & \\ 0 & \star & 0 & \star & & \\ \star & & & . & . & \\ . & & & & . & . \end{pmatrix}. \qquad (13)$$

The symbol $\star$ denotes unspecified matrix elements. We emphasize that the coefficients $(1, 1)$ and $(2, 2)$ of X are identical. This coefficient is a matrix χ which is a linear combination of various products, each product having p factors, and each factor being either a D or an E.

Using the expressions for A_2, A_3 and X, we find

$$A_2 X A_3 = \begin{pmatrix} D\chi E - E\chi D & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ . & & & . & . & \\ . & & & & . & . \end{pmatrix}. \qquad (14)$$

The product $A_2 X A_3$ has only one non-zero coefficient $D\chi E - E\chi D$ where χ is a linear combination of products of p factors D and E. Therefore, we need to show that if M is any product of p factors D and E, the matrix $DME - EMD$ is finite of size $(p + 2)$ at the most. This is achieved by induction on p and by using the explicit representations of D and E given in (9).

For $p = 0$, $M = 1$ we obtain $DE - ED = A$ which is a matrix of size 1.

Now suppose that our assertion is true for $(p - 1)$. Then, let the matrix M be a product of p factors D and E; if $M = DM_1$ (the case $M = EM_1$ is similar), we have

$$DME - EMD = D(DM_1E - EM_1D) + (DE - ED)M_1D.$$

By the induction hypothesis, $DM_1E - EM_1D$ is finite of size $(p+1)$, and multiplying it by D will increase its size by 1, at most (one verifies this by using the explicit representation given in (9)). The operator $(DE - ED)M_1D$ is equal to AM_1D and is of size less than or equal to $(p + 2)$.

We have shown that the factor A_2XA_3 in (12) is of finite size. Multiplying it on the left or on the right by any of the operators A_1, A_2, A_3 and A_4 given by (10) does not alter this property since the A_k are composed of D and E and have only a finite number of non-zero coefficients in any line and any column. This proves that the matrix $(A_{\tau_1} \dots A_{\tau_L})$ has a finite trace and that the stationary probabilities given by (6) are well defined.

4.3. *Proof of the Ansatz*

We shall use the technique developed in [30] (see, for example, [19, 31] for details). We represent the collection of the (unnormalized) stationary weights $p(\mathcal{C})$ as a state vector

$$|p\rangle = \mathrm{Tr}(A^{\otimes L}) \qquad (15)$$

where $\otimes$ denotes the tensor product and A is a column vector, having matrices as entries:

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}. \qquad (16)$$

This allows us to interpret the Markov equation (5) as a stationary Schrödinger equation with the non-Hermitian 'Hamiltonian' M:

$$M|p\rangle = \sum_{i=1}^{L} m_{i,i+1}|p\rangle = 0. \tag{17}$$

The matrix $m_{i,i+1}$ is local and acts only on the ith and the $(i+1)$th element of the tensor product in (15). We show that $m_{i,i+1}[A \otimes A]$ is a divergence-like term, i.e. there exists a vector $\hat{A}$ defined analogously to A

$$\hat{A} = \begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{pmatrix} \tag{18}$$

such that

$$m_{i,i+1}[A \otimes A] = A \otimes \hat{A} - \hat{A} \otimes A. \tag{19}$$

Summation over i leads to a global cancellation, thereby proving that the Markov equation (17) is satisfied. The proof rests upon finding four matrices, $\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$, that satisfy equation (19). In the appendix, we write the 16 quadratic equations that couple the A_κ and $\hat{A}_\kappa$ (see equations (A.1)–(A.3). An explicit representation of the $\hat{A}_\kappa$ that solves these equations is given below (here $\mathbf{1}$ denotes the identity matrix):

$$\begin{aligned}
\hat{A}_1 &= \begin{pmatrix} D/2+\mathbf{1} & 0 & E/2-\mathbf{1} & 0 & . \\ 0 & D/2+\mathbf{1} & 0 & E/2-\mathbf{1} & \\ 0 & 0 & D/2+\mathbf{1} & 0 & . \\ 0 & 0 & 0 & D/2+\mathbf{1} & . \\ . & . & . & . & . \end{pmatrix} \\
\hat{A}_2 &= \begin{pmatrix} \mathbf{1}-D/2 & \mathbf{1}-E/2 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix} \\
\hat{A}_3 &= \begin{pmatrix} E/2-\mathbf{1} & 0 & 0 & 0 & . \\ \mathbf{1}-D/2 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix} \\
\hat{A}_4 &= -\begin{pmatrix} E/2+\mathbf{1} & 0 & 0 & 0 & . \\ 0 & E/2+\mathbf{1} & 0 & 0 & . \\ D/2-\mathbf{1} & 0 & E/2+\mathbf{1} & 0 & . \\ 0 & D/2-\mathbf{1} & 0 & E/2+\mathbf{1} & . \\ . & . & . & . & . \end{pmatrix}.
\end{aligned} \tag{20}$$

Remark. There is one subtlety involved here. One should not only verify the cancellation mechanism formally but also make sure that all the traces of all the products in (19) exist. In all cases but one, this follows from section 4.2. However, the case when the last factor of the trace is A_2A_3 needs extra care because the dynamics permutes these two factors. One needs to prove the following relation, where Y denotes any product of the matrices A_k, $k = 1, 2, 3, 4$:

$$\mathrm{Tr}\{Y(A_3\hat{A}_2 - \hat{A}_3A_2)\} = \mathrm{Tr}\{\hat{A}_2YA_3 - A_2Y\hat{A}_3)\}. \tag{21}$$

However, this identity can be proved via reasoning similar to that of section 4.2.

4.4. Representation-free solution

Consider the following choice for the A_i and $\hat{A}_i$ operators:

$$A_1 = \mathbf{1} \otimes D + (D - \mathbf{1})^2 \otimes E \tag{22}$$

$$A_2 = A \otimes D + (\mathrm{i}A(D - \mathbf{1})) \otimes E \tag{23}$$

$$A_3 = A \otimes E + (\mathrm{i}(E - \mathbf{1})A) \otimes D \tag{24}$$

$$A_4 = \mathbf{1} \otimes E + (E - \mathbf{1})^2 \otimes D \tag{25}$$

and

$$\hat{A}_1 = \mathbf{1} \otimes (\mathbf{1} + D/2) + (D - \mathbf{1})^2 \otimes (E/2 - \mathbf{1}) \tag{26}$$

$$\hat{A}_2 = A \otimes (\mathbf{1} - D/2) + (\mathrm{i}A(D - \mathbf{1})) \otimes (E/2 - \mathbf{1}) \tag{27}$$

$$\hat{A}_3 = A \otimes (E/2 - \mathbf{1}) + (\mathrm{i}(\mathbf{1} - E)A) \otimes (D/2 - \mathbf{1}) \tag{28}$$

$$\hat{A}_4 = -\{\mathbf{1} \otimes (\mathbf{1} + E/2) + (E - \mathbf{1})^2 \otimes (D/2 - \mathbf{1})\} \tag{29}$$

where again $DE = E + D$, $A = DE - ED$ (projector) and where $\mathrm{i} = \sqrt{-1}$. It is then possible to show that these operators solve (A.1)–(A.3) and that, assuming a third-class particle at site L, all the traces in (6) and (19) are real and finite. The calculation is very similar to the one presented in the preceding section and will therefore be omitted here. We believe that this representation-free solution will help to generalize our solution to the case of a model with N types of particle.

4.5. Algebraic properties and recursion relations

Using our solution, it is straightforward to derive certain algebraic properties of the A_i operators and, therefore, to find recursion relations between systems of size L and $(L - 1)$. Some of these relations are listed in appendix B. In fact, we found the A_i operators by solving the model for small system sizes on the computer, guessing recursion relations and constructing suitable operators which fulfilled these relations. We want to remark that it seems extremely unlikely to us that a solution could have been constructed just by inspection of equations (A.1)–(A.3) of appendix A. However, these equations turned out to be very useful for proving that the weights given in terms of A_i are indeed the stationary weights.

4.6. Finite-size cut-off

We have shown that the matrix Ansatz using the operators given in (10) is well defined and satisfies the master equation. How can the representation be used for actual computations for systems of size L? The proof of the finiteness of the trace (section 4.2) provides a method to numerically compute the weight of any configuration of size L without involving infinite matrices. We showed that all the matrices used to evaluate the trace are of size L at most. Therefore, the operators A_κ, D and E can be restricted to a finite size Λ, with Λ large enough to ensure that the $L \times L$ matrices needed to calculate the weights are the same as those obtained by multiplying infinite-dimensional matrices. Such a cut-off procedure is possible due to the bidiagonal structure of A_κ and of D and E. For example, if we limit D and E to a finite size N and consider a product of p such matrices, the $(N - p - 1)$ first rows and columns of the resultant matrix will be the same as those obtained by taking the product of the initial infinite-dimensional matrices. To compute the weights of systems of size less than L, we must take $\Lambda > 2L$. The computation time using the matrix Ansatz grows algebraically with the system size, whereas the increase is exponential for solving the master equation. Thus our solution allows an exact numerical study of such systems for large sizes [27].

To determine the currents and the density profiles, one needs the normalization factor Z. Using exact results for systems of sizes up to 8, we guessed the following formula for Z for the case when there is only one particle of the third type ($n_3 = 1$) and $n_1, n_2, n_4 \neq 0$:

$$Z = \frac{1}{L}\binom{L}{n_1}\binom{L}{n_1+n_2}\binom{L}{n_1+n_2+1}.$$

5. Conclusion

We have studied a generalization of the asymmetric exclusion process to a system with three classes of particles and holes. This model can be mapped to an integrable two-dimensional vertex model of equilibrium statistical physics [10], but the Bethe Ansatz does not allow a simple determination of the stationary state of the ASEP. However, the stationary weights can be calculated via a matrix-product Ansatz. Although analytical formulae may be difficult to derive (the computation of the constant Z will require calculations similar to those of the diffusion constant of an open system [29]) the matrix Ansatz enables a much faster exact numerical computation of the stationary state of finite-size systems.

Our main interest is theoretical. We wanted to know what kind of matrix Ansatz (if any) would appear in multi-species processes. Some authors [24, 25, 32, 33] have used generalized quadratic algebras to study systems with many species. Associativity [25] and the finite trace condition [33] impose severe restrictions on the rates of exchange between different types of particles. The simple rates we choose (2) do not satisfy these limitations. Hence the A_κ operators we have found do not satisfy quadratic identities, but rather can be obtained as elements of the tensor product of two quadratic algebras. As emphasized in section 4.5 and in appendix B, the identities that are satisfied by the A_κ matrices can be cubic, quartic or of any higher order. We believe that the tensor structure we have obtained is fairly general. If for some special choices of the transition rates in (2), the matrices D and E can be taken to be scalars [14], the matrices given in (10) will satisfy quadratic relations.

There is a recursive structure when one adds new types of particles. The exclusion process with only one class of particles is solved by taking the matrices representing holes and particles to be both equal to 1. For two classes, the matrices D, E and A are infinite-dimensional matrices with coefficients of 1. For the three classes problem, the matrices given in (10) are infinite-dimensional with D and E as coefficients.

It is therefore natural to define a generalization of the exclusion process for N types of particles, with a priority rule such that a particle of type n can overtake a particle of type m if and only if $n < m$. This model is still integrable, and some exact results can be obtained via a Bethe Ansatz. Besides, from numerical solution of small systems one finds many relations between the rates and the matrices representing each type of particle [34]. We hope that our solution will help to find a solution for this generalized problem.

We have only studied the totally asymmetric exclusion process, it may be interesting to try to solve the partially asymmetric exclusion process where all the rules in (2), such as '$12 \rightarrow 21$ with rate 1', are modified as follows:

$$\begin{aligned} &12 \rightarrow 21 \text{ with rate } p \\ &21 \rightarrow 12 \text{ with rate } q \end{aligned} \tag{30}$$

with $p+q = 1$. We believe that a suitable tensor product structure should allow one to compute the ground state of this model. Since such a model could presumably display spontaneous symmetry breaking [35], this would be of special interest.

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Appendix A. Explicit form of the local Markov matrix

The local Markov matrix $m_{i,i+1}$ that describes the updating of a bond $(i, i+1)$ is given in the canonical basis (11), (12), (13), (14), (21), (22), ..., (44) by a 16×16 matrix:

$$
m_{i,i+1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The local divergence condition (19) translates into the following 16 coupled quadratic equations:

$$A_i A_j = \hat{A}_i A_j - A_i \hat{A}_j \qquad \text{for} \quad i < j \tag{A.1}$$

$$A_i A_j = A_j \hat{A}_i - \hat{A}_j A_i \qquad \text{for} \quad i < j \tag{A.2}$$

$$0 = A_i \hat{A}_i - \hat{A}_i A_i \qquad \text{for all} \quad i \tag{A.3}$$

where $1 \leqslant i, j \leqslant 4$.

Appendix B. Algebraic properties and recursion relations

The matrix algebra method is a way to encode recursion relations between stationary probabilities of systems of size L and $(L-1)$. For some simple models, the matrices can be constructed using ‘empirical’ recursion relations observed on exact solutions for small systems. In our case, a complete set of such relations between size L and size $(L-1)$ is difficult to obtain. However, the matrices A_1, A_2, A_3 and A_4 given in (10) satisfy a number of algebraic identities that allow one to deduce some recursions between system of different sizes. We now describe some of the relations satisfied by the matrices A_k.

(1) The matrices A_1 and A_4 satisfy the identity that was found in [29] which was used to compute the diffusion constant for an open system:

$$A_1 A_4^{p-1}(A_4 A_1 - A_1 A_4) A_1^{q-1} A_4 = A_4^p A_1^q A_4 - A_1 A_4^{p-1} A_1^q A_4 - A_1 A_4^p A_1^{q-1} A_4 + A_1 A_4^p A_1^q \tag{B.1}$$

where p and q are strictly positive integers.

(2) Some relations reduce the system size and are reminiscent of the $DE = D + E$ identity in (8). However, the following relations are cubic and not quadratic:

$$\begin{aligned} A_2 A_2 A_4 &= A_2 A_2 + A_2 A_4 \\ A_1 A_3 A_3 &= A_1 A_3 + A_3 A_3. \end{aligned} \tag{B.2}$$

(3) Other relations are similar to the second and the third equality in (8):

$$\begin{aligned} A_2 A_2 A_3 &= A_2 A_3 \\ A_2 A_3 A_3 &= A_2 A_3 \\ A_2 A_3 A_2 A_3 &= A_2 A_3. \end{aligned} \tag{B.3}$$

This last equality shows that the operator $(A_2 A_3)$ is a projector as we noted in (11).

(4) Some rules transform some particles into others without reducing the size of the system:

$$\begin{aligned} A_1 A_2 &= A_2 A_2 \\ (A_1 A_4 - A_4 A_1) A_2 &= A_2 A_4 A_2 \\ A_3 A_4 &= A_3 A_3 \\ A_3 (A_1 A_4 - A_4 A_1) &= A_3 A_1 A_3 \\ A_3 A_2 A_4 &= A_3 A_2 A_3 + A_3 A_3 A_2 \\ A_1 A_3 A_2 &= A_2 A_3 A_2 + A_3 A_2 A_2. \end{aligned} \tag{B.4}$$

We emphasize that the identities (B.1)–(B.4) are just a *subset* of the complete set of relations satisfied by the A_k operators that would be needed to give an abstract characterization of the algebra generated by A_k via generators and relations.

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